Solutions to Assignment 2

1. Find the Fourier series of the function $|\sin x|$ on $[-\pi, \pi]$.

   Solution. The function $|\sin x|$ is even. Using formulas such as

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx,$$

   we get

$$a_n = -\frac{2}{\pi}\frac{(-1)^n + 1}{n^2 - 1}, \ n \geq 1, \ a_0 = \frac{2}{\pi} \ ,$$

   and

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right) \ .$$

2. Show that

$$x^2 \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

   for $x \in [0, 2\pi]$. Here we extend x^2 which is originally defined on $[0, 2\pi]$ as a 2π-periodic function on $\mathbb{R}$. Compare it with 4(a) in Assignment 1. Compare it with 4(a) in Assignment 1.

   Solution It shows that a function may have two different Fourier expansions over a subinterval. Here we have two such expansions over $[0, \pi]$.

   Consider the function $f(x) = x^2$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\frac{x^3}{3}\bigg|_0^2 = \frac{4\pi^2}{3},$$

   and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx dx \\ &= \frac{1}{n\pi}x^2 \sin nx\bigg|_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} x \sin nx dx \\ &= \frac{2}{n^2\pi}x \cos nx\bigg|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \cos nx dx \\ &= \frac{4}{n^2}, \end{aligned}$$

   and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx dx \\ &= -\frac{1}{n\pi}x^2 \cos nx\bigg|_0^{2\pi} + \frac{2}{n\pi}\int_0^{2\pi} x \cos nx dx \\ &= -\frac{4\pi}{n} + \frac{2}{n^2\pi}x \sin nx\bigg|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \sin nx dx \\ &= -\frac{4\pi}{n}. \end{aligned}$$

3. This is an optional problem.

(a) Assume that the Fourier coefficients of a continuous, 2π-periodic function vanish identically. Show that this function must be the zero function. Hint: WLOG assume $f(0) > 0$. Use the relation

$$\int_{-\pi}^{\pi} f(x)p(x)dx = 0,$$

where $p(x)$ is a trigonometric polynomial of the form $(\varepsilon + \cos x)^k$ for some small ε and large $k > 0$.

(b) Use the result in (a) to show that if the Fourier series of a continuous, 2π-periodic function converges uniformly, then it converges uniformly to the function itself.

(c) Apply (b) to Problem 4(a) to show

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Solution

(a) Let $p_{\varepsilon,k}(x) = C_k(\varepsilon+\cos x)^{2k}$, where $C_k^{-1} = \int_{-\pi}^{\pi}(\varepsilon+\cos x)^{2k}dx$. Using $\int_{-\pi}^{\pi} p_{\varepsilon,k}(x)dx = 1$, one has

$$\int_{-\pi}^{\pi} f(x)p_{\varepsilon,k}(x)dx - f(0) = \int_{-\pi}^{\pi} p_{\varepsilon,k}(x)(f(x) - f(0))dx$$
$$= \int_{-\delta}^{\delta} p_{\varepsilon,k}(x)(f(x) - f(0))dx + (\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi})[p_{\varepsilon,k}(x)(f(x) - f(0))]dx.$$

Given $\eta > 0$, by continuity, there exists $\delta > 0$ such that

$$|f(x) - f(0)| \le \frac{1}{2}\eta, \quad \forall x \in [-\delta, \delta].$$

Then

$$\left|\int_{-\delta}^{\delta} p_{\varepsilon,k}(x)(f(x) - f(0))dx\right| \le \frac{1}{2}\eta \int_{-\delta}^{\delta} p_{\varepsilon,k}(x)dx \le \frac{1}{2}\eta \int_{-\pi}^{\pi} p_{\varepsilon,k}(x)dx \le \frac{1}{2}\eta,$$

and

$$|(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi})[p_{\varepsilon,k}(x)(f(x) - f(0))]dx| \le \max_{\delta \le |x| \le \pi} p_{\varepsilon,k}(x) \int_{-\pi}^{\pi} |f(x) - f(0)|dx$$

First choose ε sufficiently small so that for $\delta \le |x| \le \pi, |(\epsilon + \cos x)| < 1$, then choose k sufficiently large so that

$$\max_{\delta \le |x| \le \pi} p_{\varepsilon,k}(x) \le \frac{1}{1 + \int_{-\pi}^{\pi} |f(x) - f(0)|dx} \times \frac{\eta}{2}.$$

Hence one has

$$\left|\int_{-\pi}^{\pi} p_{\varepsilon,k}(x)(f(x) - f(0))dx\right| \le \eta.$$

Since $\eta > 0$ is arbitrary, this shows $\int_{-\pi}^{\pi} f(x)p_{\varepsilon,k}(x)dx \to f(0)$, for suitably chosen $\varepsilon \to 0, k \to \infty$. By Problem 5, the trigonometric polynomial $p_{\varepsilon,k}(x)$ is a finite Fourier series, which can be written as $p_{\varepsilon,k}(x) = \sum_{n=-k}^{k} c_n e^{inx}$. Suppose the Fourier coefficients of $f(x)$ vanish identically, then $\int_{-\pi}^{\pi} f(x)p_{\varepsilon,k}(x)dx = 0$. This implies that $f(0) = 0$. By translations, it holds that for $x \in [-\pi, \pi], f(x) = 0$. Also refer to Stein-Shakarchi, Fourier Analysis, page 39-40 for a slightly variant proof.

(b) Suppose $S_N(x) = a_0 + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx)$ is the partial sums of the Fourier series of a continuous, 2π-periodic function u, and that S_N converges uniformly. Let $v(x) = \lim_{N\to\infty} S_N(x)$. By the uniform convergence, $v(x)$ is a continuous, 2π-periodic function. Furthermore,

$$\int v(x)\cos nx\,dx = \lim_{N\to\infty} \int a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)\cos nx\,dx = \pi b_n,$$

and the same holds for a_n. Let $w = u - v$. Then the Fourier coefficients of w vanish identically, and by (a) one has $w \equiv 0$. Hence $u = v$ and that S_N converge uniformly to u.

(c) As we knew, $\dfrac{\pi^2}{3} - 4\sum_{n=1}^\infty \dfrac{(-1)^{n+1}}{n^2} \cos nx$ is the Fourier series of the function $u(x) = x^2$, and the series converges uniformly. Hence by (b), it must converge to $u(x) = x^2$. One has

$$0 = u(0) = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos 0,$$

and we obtain

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Remark. Part (a) implies that when two 2π-periodic, continuous functions have the same Fourier series, they must be identical. Such result follows from the Uniqueness Theorem in the Notes. For, by this theorem it follows that these two functions are the same away from a set of measure. In other words,

$$\int_{-\pi}^{\pi} |f(x) - g(x)|\,dx = 0\ .$$

However, since f and g are continuous, f must equal to g everywhere. This exercise, however, gives an independent proof of this fact (without using the Uniqueness Theorem).

4. Let f be a complex valued 2π-periodic function whose derivative is again integrable on $[-\pi, \pi]$. Show that c_n and c_n', the Fourier coefficients of f and f' respectively, satisfies the relation $c_n' = inc_n, n \in \mathbb{Z}$. Do not do it formally. Use the definition of the integration of complex valued functions.

Solution. Here we integrate by parts over a period. (The proof given in class was kind of formal.)

$$\begin{aligned}
c_n' &\equiv \int f'(x)e^{-inx}dx \\
&= \int (f_1'(x) + if_2(x))(\cos nx - i\sin nx)dx \\
&= \int (f_1'(x)\cos nx + f_2'(x)\sin nx)dx + i\int(-f_1'(x)\sin nx + f_2'(x)\cos nx)dx \\
&= n\int (f_1(x)\sin nx - f_2(x)\cos nx)dx + in\int (f_1(x)\cos nx + f_2(x)\sin nx)dx \\
&= n\int (f_1(x) + if_2(x))\sin nx dx + in\int (f_1(x) + if_2(x))\cos nx dx \\
&= ni\int (f_1(x) + if_2(x))(\cos nx - i\sin nx)dx \\
&\equiv nic_n\ .
\end{aligned}$$

5. Let $C^\infty_{2\pi}$ be the class of all smooth 2π-periodic, complex-valued functions and $\mathcal{C}^\infty$ the class of all complex bisequences satisfying $c_n = \circ(n^{-k})$ as $n \to \pm\infty$ for every k. Show that the Fourier transform $f \mapsto \hat{f}$ is bijective from $C^\infty_{2\pi}$ to $\mathcal{C}^\infty$.

   Solution First, we show that the Fourier coefficients of a smooth, periodic function are rapidly decreasing. A repeated application of Problem 1 shows that $(in)^k\hat{f}(n)$ is equal to the Fourier coefficients of $f^{(k)}$ for every k. In general, we have

$$\begin{aligned} |\hat{g}(n)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\{\pi} g(x)e^{-inx}\ dx\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)||e^{-inx}|dx \\ &\le \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} |g(x)|dx\right| \equiv M(g)\ , \end{aligned}$$

   that is, the Fourier coefficients of any integrable function are always uniformly bounded. Now, for a fixed k, we have

$$|\hat{f}(n)| = \left|\frac{1}{(in)^k}\hat{f^{(k)}}(n)\right| \le \frac{M(f^{(k)})}{n^k}\ ,$$

   so $\{\hat{f}(n)\}$ belongs to $\mathcal{C}^\infty$.

   Second, onto. Let $\{c_n\}$ be a rapidly decreasing bisequence. Define

$$f(x) \equiv \sum_{-\infty}^{\infty} c_n e^{inx}\ .$$

   Taking $k = 2$, we have

$$\left|c_n e^{inx}\right| = |c_n| \le \frac{C}{n^2},$$

   for some constant C. By M-Test the right hand side in f is a uniformly convergent series of functions so f is well-defined. Furthermore, as uniform convergence preserves continuity, f is also continuous. By using M-Test to $\sum_{-\infty}^{\infty} inc_n e^{inx}$ (taking $k = 3$), we see that it is also uniformly convergent. By one exchange theorem we learned in 2060 we conclude that f is differentiable and $f'(x) = \sum_{-\infty}^{\infty} inc_n e^{inx}$. Repeating this argument we see that $f \in C^\infty_{2\pi}$.

   Third, one-to-one. By Theorem 1.7 $f(x) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx}$ and $g(x) = \sum_{-\infty}^{\infty} \hat{g}(n)e^{inx}$. When $\hat{f}(n) = \hat{g}(n)$, it is obvious that $f \equiv g$.

6. Propose a definition for $\sqrt{d/dx}$. This operator should be a linear map which maps $C^\infty_{2\pi}$ to itself satisfying

$$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}}f = \frac{d}{dx}f,$$

   for all smooth, 2π-periodic f.

   Solution Use complex notation. For a smooth function f,

$$\widehat{f'}(n) = in\hat{f}(n). \tag{1}$$

   In view of $i = e^{i\pi/2}$, this motivates us to define $g(x) = \sqrt{d/dx}f(x)$ to be the function whose Fourier series is given by

$$\hat{g}(n) = c_n = e^{i\pi/4}\sqrt{n}\hat{f}(n).$$

That is,

$$g(x)=\sum_{n=-\infty}^{\infty} e^{i\pi/4}\sqrt{n}\hat{f}(n)e^{inx}\ .$$

When $f\in C^\infty_{2\pi}$, by Problem 5 in Assignment 1 (see also the previous problem), it is easy to see that the series in the right hand side of g defines again a smooth and 2π-periodic function, and the convergence is uniform. Hence $\sqrt{d/dx}$ is a linear map on $C^\infty_{2\pi}$ to itself. Writing $h(x)=\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}}f(x)$, then

$$\hat{h}(n)=e^{i\pi/4}\sqrt{n}\hat{g}(n)=e^{i\pi/4}\sqrt{n}e^{i\pi/4}\sqrt{n}\hat{f}(n)=(in)\hat{f}(n).$$

By the the uniqueness of the Fourier series, one has

$$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}}f=\frac{d}{dx}f.$$

This problem demonstrates the power of Fourier series. It is hopeless to define fractional derivative on the function directly.

7. Let f be a continuous, 2π-periodic function and its primitive function be given by

$$F(x)=\int_0^x f(x)dx.$$

Show that F is 2π-periodic if and only if f has zero mean. In this case,

$$\hat{F}(n)=\frac{1}{in}\hat{f}(n),\quad \forall n\neq 0.$$

Solution. From

$$\begin{aligned}F(x+2\pi) &= \int_0^{x+2\pi} f(y)\ dy\\ &= \int_0^{2\pi} f(y)\ dy+\int_{2\pi}^{x+2\pi} f(y)\ dy\\ &= \int_0^{2\pi} f(y)\ dy+\int_0^{x} f(y)\ dy\\ &= \int_0^{2\pi} f(y)\ dy+F(x)\ ,\end{aligned}$$

it is clear that F is of period 2π if and only if f has zero mean. The formula comes by easily.

8. Let $\mathcal{C}'$ be the subspace of $\mathcal{C}$ consisting of all bisequences $\{c_n\}$ satisfying $\sum_{-\infty}^{\infty}|c_n|^2<\infty$.

(a) For $f\in R[-\pi,\pi]$, show that

$$2\pi\sum_{-\infty}^{\infty}|c_n|^2\leq\int_{-\pi}^{\pi}|f|^2\ .$$

(b) Deduce from (a) that the Fourier transform $f\mapsto\hat{f}(n)$ maps $R_{2\pi}$ into $\mathcal{C}'$.

(c) Explain why the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha}~, \quad \alpha \in (0, 1/2]~,$$

is not the Fourier series of any function in $R_{2\pi}$.

Solution. (a) Using $\left(f(x) - \sum_{k=-n}^{n} c_k e^{ikx}\right) \overline{\left(f(x) - \sum_{k=-n}^{n} c_k e^{ikx}\right)} \geq 0$ for all n and x,

$$\begin{aligned}
0 &\leq \int (f(x) - \sum_{k=-n}^{n} c_k e^{ikx}) \overline{(f(x) - \sum_{k=-n}^{n} c_k e^{ikx})}\, dx \\
&= \int (f(x) - \sum_{k=-n}^{n} c_k e^{-ikx})(\overline{f(x)} - \sum_{j=-n}^{n} \overline{c_j} e^{-ijx})\, dx \\
&= \int (|f(x)|^2 - \sum_{j=-n}^{n} f(x)\overline{c_j} e^{-ijx} - \sum_{k=-n}^{n} \overline{f(x)} c_k e^{ikx} + \sum_{j,k=-n}^{n} c_j \overline{c_k} e^{i(j-k)x})\, dx \\
&= \int (|f(x)|^2 - 2\pi \sum_{k=-n}^{n} |c_k|^2)\, dx~,
\end{aligned}$$

by the orthogonality of e^{-ikx}'s. The desired inequality follows by letting n go to infinity.

(b) It is clear from (a).

(c) From $\sum |c_n|^2 < \infty$ one deduces that $\sum a_n^2, \sum b_n^2 < \infty$ also hold when the function is of real-valued. Now, if the given trigonometric series come from an integrable function, then $\sum a_n^2 = \sum \dfrac{1}{n^{2\alpha}}$ must be finite. But now it is not when $\alpha \in (0, 1]$. We conclude that it is not a Fourier series.

9. Let f be a continuous piecewise C^1 function in $C_{2\pi}$. In other words, there exist $-\pi = a_1 < a_2 < \cdots < a_N = \pi$ and C^1-functions f_j defined on $[a_j, a_{j+1}],\ j = 0, \cdots, N-1$ such that $f = f_j$ on (a_j, a_{j+1}). Show that its Fourier series converges uniformly to itself. Hint: Let $M = \max_j \{\sup |f_j'(x)| :\ x \in [a_j, a_{j+1}]\}$. Establish $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in [-\pi, \pi]$.

Solution Let f be a C^1-piecewise, continuous 2π-periodic function. There exist $-\pi = a_1 < a_2 < \cdots < a_N = \pi$ and C^1-functions f_j defined on $[a_j, a_{j+1}],\ j = 0, \cdots, N-1$ such that $f = f_j$ on (a_j, a_{j+1}). Let $M = \max_j \{\sup |f_j'(x)| :\ x \in [a_j, a_{j+1}]\}$. For $x, y \in [-\pi, \pi]$ with $x \leq a_{j_m} < \cdots \leq a_{j_M} \leq y$, then

$$\begin{aligned}
|f(y) - f(x)| &= |f(y) - f(a_{j_M}) + \sum_{j=j_m}^{j_M - 1} (f(a_{j+1}) - f(a_j)) + f(a_{j_m}) - f(x)| \\
&= |f_{j_M}(y) - f_{j_M}(a_{j_M}) + \sum_{j=j_m}^{j_M - 1} (f_j(a_{j+1}) - f_j(a_j)) + f_{j_m - 1}(a_{j_m}) - f_{j_m - 1}(x)| \\
&\leq M|y - a_{j_M}| + M \sum_{j=j_m}^{j_M - 1} |a_{j+1} - a_j| + M|a_{j_m} - x| \\
&= M|y - x|~.
\end{aligned}$$

Thus $f(x)$ is uniformly Lipschitz continuous. Now uniform convergence follows from Theorem 1.7.

Note The Fourier series of a piecewise C^1-, continuous 2π-periodic function converges uniformly to the function itself is the most commonly used criterion for convergence.

10. Show that for a Lipschitiz continuous, 2π-periodic function, its Fourier coefficients satisfy

$$|a_n| \leq \frac{C\pi}{n}, \quad |b_n| \leq \frac{C\pi}{n} \ ,$$

for some constant C.

Solution Since f is 2π-periodic, one has

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ &= \frac{1}{\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(x+\frac{\pi}{n})\sin n(x+\frac{\pi}{n}) dx \\ &= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})\sin nx dx. \end{aligned}$$

Hence by the assumption on f, one has $|f(x+y)-f(x)| \leq L|x-y|, \forall x, y \in [-\pi,\pi]$, and

$$\begin{aligned} |b_n| &= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - f(x+\frac{\pi}{n}))\sin nx dx \right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f(x+\frac{\pi}{n})||\sin nx| dx \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} L\frac{\pi}{n} dx \\ &\leq L\frac{\pi}{n}. \end{aligned}$$

Similar estimates hold for a_n's.